Matrik – Determinan dengan Kofaktor

Matrik berordo 2x2

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\rangle\rangle\ |A| = a_{11}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}|a_{22}| - a_{12}|a_{21}| = a_{11}a_{22} - a_{12}a_{21}$$

$$\rangle\rangle\ |A| = -a_{21}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -a_{21}|a_{12}| + a_{22}|a_{11}| = a_{11}a_{22} - a_{12}a_{21}$$

$$\rangle\rangle\ |A| = a_{11}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{21}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}|a_{22}| - a_{21}|a_{12}| = a_{11}a_{22} - a_{12}a_{21}$$

$$\rangle\rangle\ |A| = -a_{12}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -a_{12}|a_{21}| + a_{22}|a_{11}| = a_{11}a_{22} - a_{12}a_{21}$$

Matrik berordo 3x3

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Berdasarkan baris ke-1

$$\Rightarrow |A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

Berdasarkan kolom ke-2

$$\Rightarrow |A| = a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{32}(a_{13}a_{21} - a_{11}a_{23})$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{12}(-1)^{1+2}M_{12} + a_{22}(-1)^{2+2}M_{22} + a_{32}(-1)^{3+2}M_{32}$$

Demikian seterusnya, boleh memilih baris atau kolom mana saja sehingga diperoleh hitungan yang paling mudah, mungkin karena dalam baris atau kolom tersebut memuat banyak angka 0 atau 1.

Ini adalah cara dalam menentukan determinan matrik berordo 4x4, 5x5, dan seterusnya. Metode perkalian elemen (aturan Sarrus's) seperti pada matrik berordo 2x2 dan 3x3 tidak bisa dipakai lagi.

Untuk A matrik berordo $n \times n$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

$$= a_{i1}(-1)^{i+1} M_{i1} + a_{i2}(-1)^{i+2} M_{i2} + a_{i3}(-1)^{i+3} M_{i3} + \dots + a_{ij}(-1)^{i+j} M_{ij} + \dots + a_{in}(-1)^{i+n} M_{in}$$

(berdasar baris ke - i)

$$= a_{1j}(-1)^{1+j} M_{1j} + a_{2j}(-1)^{2+j} M_{2j} + a_{3j}(-1)^{3+j} M_{3j} + \dots + a_{ij}(-1)^{i+j} M_{ij} + \dots + a_{nj}(-1)^{n+j} M_{nj}$$

(berdasar kolom ke - j)

Contoh:

Tentukan determinan matrik $\text{A} = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 1 & 6 \\ 2 & 4 & 7 \end{bmatrix}$ dan $\text{B} = \begin{bmatrix} 5 & 2 & 2 & 6 \\ 0 & 1 & 1 & 1 \\ 7 & 4 & 3 & 5 \\ 8 & 6 & 2 & 3 \end{bmatrix}$.

Jawab:

$$|\text{A}| = \begin{vmatrix} 1 & 3 & 5 \\ 4 & 1 & 6 \\ 2 & 4 & 7 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 6 \\ 4 & 7 \end{vmatrix} + 4 \cdot (-1)^{2+1} \begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} + 2 \cdot (-1)^{3+1} \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = -17 - 4 + 26 = 5$$

$$|\text{B}| = \begin{vmatrix} 5 & 2 & 2 & 6 \\ 0 & 1 & 1 & 1 \\ 7 & 4 & 3 & 5 \\ 8 & 6 & 2 & 3 \end{vmatrix}$$

$$= 0 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 2 & 6 \\ 4 & 3 & 5 \\ 6 & 2 & 3 \end{vmatrix} + 1 \cdot (-1)^{2+2} \begin{vmatrix} 5 & 2 & 6 \\ 7 & 3 & 5 \\ 8 & 2 & 3 \end{vmatrix} + 1 \cdot (-1)^{2+3} \begin{vmatrix} 5 & 2 & 6 \\ 7 & 4 & 5 \\ 8 & 6 & 3 \end{vmatrix} + 1 \cdot (-1)^{2+4} \begin{vmatrix} 5 & 2 & 2 \\ 7 & 4 & 3 \\ 8 & 6 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 2 & 6 \\ 7 & 3 & 5 \\ 8 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 5 & 2 & 6 \\ 7 & 4 & 5 \\ 8 & 6 & 3 \end{vmatrix} + \begin{vmatrix} 5 & 2 & 2 \\ 7 & 4 & 3 \\ 8 & 6 & 2 \end{vmatrix}$$

$$= 2\cdot(-1)^{1+2}\begin{vmatrix}7 & 5\\ 8 & 3\end{vmatrix} + 3\cdot(-1)^{2+2}\begin{vmatrix}5 & 6\\ 8 & 3\end{vmatrix} + 2\cdot(-1)^{3+2}\begin{vmatrix}5 & 6\\ 7 & 5\end{vmatrix}$$

$$-5\cdot(-1)^{1+1}\begin{vmatrix}4 & 5\\ 6 & 3\end{vmatrix} - 2\cdot(-1)^{1+2}\begin{vmatrix}7 & 5\\ 8 & 3\end{vmatrix} - 6\cdot(-1)^{1+3}\begin{vmatrix}7 & 4\\ 8 & 6\end{vmatrix}$$

$$+5\cdot(-1)^{1+1}\begin{vmatrix}4 & 3\\ 6 & 2\end{vmatrix} + 2\cdot(-1)^{1+2}\begin{vmatrix}7 & 3\\ 8 & 2\end{vmatrix} + 2\cdot(-1)^{1+3}\begin{vmatrix}7 & 4\\ 8 & 6\end{vmatrix}$$

$$= -2\cdot(21-40)+3\cdot(15-48)-2\cdot(25-42)-5\cdot(12-30)+2\cdot(21-40)-6\cdot(42-32)$$
$$+5\cdot(8-18)-2\cdot(14-24)+2\cdot(42-32)$$

$$= -2\cdot(-19)+3\cdot(-33)-2\cdot(-17)-5\cdot(-18)+2\cdot(-19)-6\cdot(10)+5\cdot(-10)-2\cdot(-10)+2\cdot(10)$$

$$= 38-99+34+90-38-60-50+20+20$$

$$= -45$$

Catatan:
M_{ij} adalah minor elemen a_{ij} pada matrik A yang didefinisikan sebagai determinan sub matrik A dengan menghapus baris ke i dan kolom ke j.
c_{ij} disebut kofaktor elemen a_{ij} pada matrik A didefinisikan dengan $c_{ij} = (-1)^{i+j} M_{ij}$.