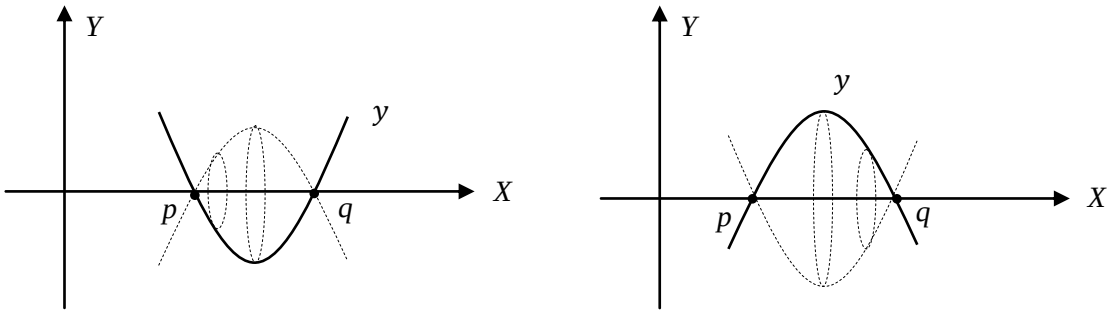
Integral – Volume Benda Putar Kurva Mengelilingi Sb-X



Misalkan $y = ax^2 + bx + c$, maka volume benda putar kurva mengelilingi sb-X adalah

$$V = \pi \int_p^q y^2 dx = \pi \int_p^q (ax^2 + bx + c)^2 dx$$

$$= \pi \int_p^q (a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2) dx$$

$$= \pi \left[\frac{1}{5}a^2x^5 + \frac{1}{2}abx^4 + \frac{1}{3}(b^2 + 2ac)x^3 + bcx^2 + c^2x\right]_p^q$$

$$= \pi \left[\frac{1}{5}a^2(q^5 - p^5) + \frac{1}{2}ab(q^4 - p^4) + \frac{1}{3}(b^2 + 2ac)(q^3 - p^3) + bc(q^2 - p^2) + c^2(q - p)\right]$$

$$= \pi \left[\frac{1}{5}a^2(q - p)(q^4 + q^3p + q^2p^2 + qp^3 + p^4) + \frac{1}{2}ab(q - p)(q + p)(q^2 + p^2) + \frac{1}{3}(b^2 + 2ac)(q - p)(q^2 + qp + p^2) + bc(q - p)(q + p) + c^2(q - p)\right]$$

$$= \pi \left[\frac{1}{5}a^2(q - p)((q + p)^4 - 3qp((q + p)^2 - 2qp) - 5(qp)^2) + \frac{1}{2}ab(q - p)(q + p)((q + p)^2 - 2qp) + \frac{1}{3}(b^2 + 2ac)(q - p)((q + p)^2 - qp) + bc(q - p)(q + p) + c^2(q - p)\right]$$

$$= \pi \left[\frac{1}{5}a^2\left(\frac{\sqrt{D}}{a}\right)\left(\left(\frac{-b}{a}\right)^4 - 3\frac{c}{a}\left(\left(\frac{-b}{a}\right)^2 - 2\frac{c}{a}\right) - 5\left(\frac{c}{a}\right)^2\right) + \frac{1}{2}ab\left(\frac{\sqrt{D}}{a}\right)\left(\frac{-b}{a}\right)\left(\left(\frac{-b}{a}\right)^2 - 2\frac{c}{a}\right) + \frac{1}{3}(b^2 + 2ac)\left(\frac{\sqrt{D}}{a}\right)\left(\left(\frac{-b}{a}\right)^2 - \frac{c}{a}\right) + bc\left(\frac{\sqrt{D}}{a}\right)\left(\frac{-b}{a}\right) + c^2\left(\frac{\sqrt{D}}{a}\right)\right]$$

$$= \frac{\sqrt{D}}{a}\pi \left[\frac{1}{5}a^2\left(\frac{b^4}{a^4} - \frac{3c}{a}\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) - \frac{5c^2}{a^2}\right) + \frac{1}{2}ab\left(\frac{-b}{a}\right)\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) + \frac{1}{3}(b^2 + 2ac)\left(\frac{b^2}{a^2} - \frac{c}{a}\right) + bc\left(\frac{-b}{a}\right) + c^2\right]$$

$$= \frac{\sqrt{D}}{a}\pi\left[\frac{1}{5}\left(\frac{a^2b^4}{a^4} - \frac{3a^2b^2c}{a^3} + \frac{6a^2c^2}{a^2} - \frac{5a^2c^2}{a^2}\right) + \frac{1}{2}\left(-\frac{ab^4}{a^3} + \frac{2ab^2c}{a^2}\right)\right.$$
$$\left. + \frac{1}{3}\left(\frac{b^4}{a^2} + \frac{2ab^2c}{a^2} - \frac{b^2c}{a} - \frac{2ac^2}{a}\right) - \frac{b^2c}{a} + c^2\right]$$

$$= \frac{\sqrt{D}}{a}\pi\left[\frac{a^2b^4 - 3a^3b^2c + a^4c^2}{5a^4} + \frac{-a^2b^4 + 2a^3b^2c}{2a^4} + \frac{a^2b^4 + a^3b^2c - 2a^4c^2}{3a^4} - \frac{a^3b^2c}{a^4} + \frac{a^4c^2}{a^4}\right]$$

$$= \frac{\sqrt{D}}{a}\pi\left[\frac{6a^2b^4 - 18a^3b^2c + 6a^4c^2}{30a^4} + \frac{-15a^2b^4 + 30a^3b^2c}{30a^4} + \frac{10a^2b^4 + 10a^3b^2c - 20a^4c^2}{30a^4}\right.$$
$$\left. - \frac{30a^3b^2c}{30a^4} + \frac{30a^4c^2}{30a^4}\right]$$

$$= \frac{\sqrt{D}}{a}\pi\left[\frac{a^2b^4 - 8a^3b^2c + 16a^4c^2}{30a^4}\right]$$

$$= \frac{\sqrt{D}}{a}\pi\left[\frac{b^4 - 8ab^2c + 16a^2c^2}{30a^2}\right]$$

$$= \frac{\sqrt{D}}{30a^3}\pi[(b^2 - 4ac)^2]$$

$$= \boldsymbol{\frac{D^2\sqrt{D}}{30a^3}\pi}$$

Karena volume selalu positif, dapatlah kita tulis: $V = \frac{D^2\sqrt{D}}{30|a^3|}\pi;\ \ D > 0$

Catatan:

1. p dan q adalah akar-akar dari $ax^2 + bx + c = 0$
2. $p, q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $p + q = -\frac{b}{a}, q - p = \frac{\sqrt{D}}{a}$, dan $pq = \frac{c}{a}$
3. $D = b^2 - 4ac$