SOLUTION TO IMO 2007

August 23, 2007

1. (a) Let $1 \le p \le q \le r \le n$ be indices for which

$$d = d_q$$

$$a_p = \max\{a_j : 1 \le j \le q\}$$

$$a_r = \min\{a_j : q \le j \le n\}$$

and thus $d = a_p - a_r$ (these indices are not necessarily unique).
For arbitrary real numbers $x_1 \le x_2 \le \cdots \le x_n$, consider just the two quantities $|x_p - a_p|$ and $|x_r - a_r|$. Since

$$(a_p - x_p) + (x_r - a_r) = (a_p - a_r) + (x_r - x_p) \ge d,$$

we have either $a_p - x_p \ge d/2$ or $x_r - a_r \ge d/2$.
Hence

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \max\{|x_p - a_p|, |x_r - a_r|\} \ge \max\{a_p - x_p, x_r - a_r\} \ge \frac{d}{2}.$$

(b) Define the sequence (x_k) as

$$x_1 = a_1 - \frac{d}{2}$$

$$x_k = \max\{x_{k-1}, a_k - \frac{d}{2}\} \text{ for } 2 \le k \le n.$$

We show that we have equality in (1) for this sequence.
By definition, sequence (x_k) is non-decreasing and $x_k - a_k \ge -d/2$ for all $1 \le k \le n$. Next we prove that $x_k - a_k \le d/2$ for all $1 \le k \le n$. Consider an arbitrary index $1 \le k \le n$. Let $l \le k$ be the smallest index such that $x_k = x_l$. We have either $l = 1$ or $l \ge 2$ and $x_l > x_{l+1}$. In both cases

$$x_k = x_l = a_l - \frac{d}{2}.$$

Since

$$a_l - a_k \le \max\{a_j : 1 \le j \le k\} - \min\{a_j : k \le j \le n\} = d_k \le d,$$

we have

$$x_k - a_k = a_l - a_k - \frac{d}{2} \le d - \frac{d}{2} = \frac{d}{2}.$$

We obtained that

$$-\frac{d}{2} \le x_k - a_k \le \frac{d}{2}$$

for all $1 \le k \le n$, so

$$\max\{|x_i - a_i| : 1 \le i \le n\} \le \frac{d}{2}.$$

We have equality because $|x_1 - a_1| = d/2$.

Alternative solution for (b)

For each $1 \le i \le n$, let

$$M_i = \max\{a_j : 1 \le j \le i\}$$
$$m_i = \min\{a_j : i \le j \le n\}.$$

For all $1 \le i \le n$, we have

$$M_i = \max\{a_1, \ldots, a_i\} \le \max\{a_1, \ldots, a_i, a_{i+1}\} = M_{i+1}$$

and

$$m_i = \min\{a_i, a_{i+1} \ldots, a_n\} \le \min\{a_{i+1} \ldots, a_n\} = m_{i+1}.$$

Therefore sequences (M_i) and (m_i) are non-decreasing. Moreover, since a_i is listed in both definitions, $m_i \le a_i \le M_i$.

To achieve equality in (1), set $x_i = \frac{M_i + m_i}{2}$. Since sequences (M_i) and (m_i) are non-decreasing, this sequence is non-decreasing as well. From $d_i = M_i - m_i$, we obtain that

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \le x_i - a_i \le x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Therefore

$$\max\{|x_i - a_i| : 1 \le i \le n\} \le \max\{\frac{d_i}{2} : 1 \le i \le n\} = \frac{d}{2}.$$

Since the opposite inequality has been proved in (a), we must have equality.

2. If $CF = CG$, then $\angle FGC = \angle GFC$, hence $\angle GAB = \angle GFC = \angle FGC = \angle FAD$, and ℓ is a bisector.

   Assume that $CF < GC$. Let EK and EL be the altitudes in the isosceles triangles ECF and EGC, respectively. Then in the right triangles EKF and ELC we have $EF = EC$ and $KF = CF/2 < GC/2 = LC$, so

   $$KE = \sqrt{EF^2 - KF^2} > \sqrt{EC^2 - LC^2} = LE.$$

   Since quadrilateral $BCED$ is cyclic, we have $\angle EDC = \angle EBC$, so right triangles BEL and DEK are similar. Then $KE > LE$ implies $DK > BL$, hence

   $$DF = DK - KF > BL - LC = BC = AD.$$

   But triangles ADF and GCF are similar, so we have $1 > AD/DF = GC/CF$; this contradicts our assumption.

   The case $CF > CG$ is completely similar. We consequently obtain the converse inequalities $KF > LC, KE < LE, DK < BL, DF < AD$, hence $1 < AD/DF = GF/CF$, a contradiction.

3. We present an algorithm to arrange the competitors. Let the rooms be A and B. We start with an initial arrangement, and then we modify it several times by sending one person to the other room. At any state of the algorithm, A and B denote the sets of the competitors in the rooms, and $c(A)$ and $c(B)$ denote the largest sizes of cliques in the rooms.

   STEP 1: Let M be one of the cliques of largest size, $|M| = 2m$. Send all members of M to A and all others to B.

   Since M is a clique of the largest size, we have $c(A) = |M| \geq c(B)$.

   STEP 2: While $c(A) > c(B)$, send one person from A to B.

   Note that $c(A) > c(B)$ implies that A is not empty. In each step, $c(A)$ decreases by 1 and $c(B)$ increases by at most 1. So at the end we have $c(A) \leq c(B) \leq c(A) + 1$.

   We also have $c(A) = |A| \geq m$ at the end. Otherwise we would have at least $m + 1$ members of M in B and at most $m - 1$ in A, implying $c(B) - c(A) \geq (m + 1) - (m - 1) = 2$.

   STEP 3: Let $k = c(A)$. If $c(B) = k$ then STOP.

   If we reached $c(A) = c(B) = k$ then we have found the desired arrangement. In all other cases we have $c(B) = k + 1$. From the estimate above we also know that $k = |A| = |A \cap M| \geq m$ and $|B \cap M| \leq m$.

   STEP 4: If there exists a competitor $x \in B \cap M$ and a clique $C \subset B$ such that $|C| = k + 1$ and $x \notin C$, then move x to A and STOP.

   After moving x back to A, we will have $k + 1$ members of M in A, thus $c(A) = k + 1$. Due to $x \notin C$, $c(B) = |C|$ is not decreased, and after this step we have $c(A) = c(B) = k + 1$.

   If there is no such competitor x, then in B all cliques of size $k + 1$ contain $B \cap M$ as a subset.

   STEP 5: While $c(B) = k + 1$, choose a clique $C \subset B$ such that $|C| = k + 1$ and move one member of $C \setminus M$ to A.

   Note that $|C| = k + 1 > m \geq |B \cap M|$, so $C \setminus M$ is not empty. Every time we move a single person from B to A, $c(B)$ decreases by at most 1. Hence, at the end of this loop we have $c(B) = k$.

   In A we have the clique $A \cap M$ with size $|A \cap M| = k$, thus $c(A) \geq k$. We prove that there is no clique of larger size there. Let $Q \subset A$ be an arbitrary clique. We show that $|Q| \leq k$.

   In A, specially in set Q, there can be two types of competitors: (i) some members of M. Since M is a clique, they are friends with all members of $|B \cap M|$; (ii) competitors which were moved to A in STEP 5. Each of them has been in a clique with $B \cap M$ so they are also friends with all members of $B \cap M$.

   Hence all members of Q are friends with all the members of $B \cap M$. Sets Q and $B \cap M$ are cliques themselves, so $Q \cup (B \cap M)$ is also a clique. Since M is a clique of the largest size,

$$|M| \geq |Q \cup (B \cap M)| = |Q| + |B \cap M| = |Q| + |M| - |A \cap M|,$$

   i.e. $|Q| \leq |A \cap M| = k$. Finally after STEP 5 we have $c(A) = c(B) = k$.

4. If $AC = BC$ then triangle ABC is isosceles, triangles RQL and RPK are symmetric about the bisector CR and the statement is trivial.

   If $AC \neq BC$ then it can be assumed without loss of generality that $AC < BC$. Denote the circumcenter by O. The right triangle CLQ and CKP have equal angles at C, so they are similar:

$$\angle CPK = \angle CQL = \angle OQP \quad \text{and} \quad \frac{QL}{PK} = \frac{CQ}{CP}.$$

   Let ℓ be the perpendicular bisector of chord CR; ℓ passes through the circumcenter O. Due to equal angles at P and Q, triangle OPQ is isosceles with $OP = OQ$. Then line ℓ is the axis of symmetry in this triangle as well. Therefore, points P and Q lie symmetrically on line segment CR, i.e. $RP = CQ$ and $RQ = CP$.

   Triangles RQL and RPK have equal angles at Q and P respectively. Then

$$\frac{|RQL|}{|RPK|} = \frac{\frac{1}{2} \cdot RQ \cdot QL \cdot \sin \angle RQL}{\frac{1}{2} \cdot RP \cdot PK \cdot \sin \angle RPK} = \frac{RQ}{RP} \cdot \frac{QL}{PK}.$$

   Using the facts already derived, we have

$$\frac{|RQL|}{|RPK|} = \frac{RQ}{RP} \cdot \frac{QL}{PK} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

   Alternative solution

   Assume again $AC < BC$. Denote the circumcenter by O and let γ be the angle at C. Similarly to the first solution, from right triangles CLQ and CKP we obtain $\angle OPQ = \angle OQP = 90^\circ - \gamma/2$. Then triangle OPQ is isosceles, $OP = OQ$ and moreover $\angle POQ = \gamma$.

   As is well known, point R is the midpoint of arc AB and $\angle ROA = \angle BOR = \gamma$.

   Consider the rotation around point O by angle γ. This transform moves A to R, R to B, and Q to P; hence triangles RQA and BPR are congruent, and they have the same area. Triangles RQL and RQA have RQ as common side, so the ratio between their areas is

$$\frac{RQL}{RQA} = \frac{d(L, CR)}{d(A, CR)} = \frac{CL}{CA} = \frac{1}{2}.$$

   ($d(X, YZ)$ denotes the distance between point X and line YZ).

   It can be obtained similarly that

$$\frac{|RPK|}{|BPR|} = \frac{CK}{CB} = \frac{1}{2}.$$

   Now the proof can be completed as

$$|RQL| = \frac{1}{2}|RQA| = \frac{1}{2}|BPR| = |RPK|.$$

5. Call a pair (a, b) of positive integers bad if $4ab - 1$ divides $(4a^2 - 1)^2$ but $a \neq b$. In order to prove that bad pairs do not exist, we present two properties of them which provide an infinite descent.

Property 1: If (a, b) is a bad pair and $a < b$ then there exists a positive integer $c < a$ such that (a, c) is also bad.

Let $r = \frac{(4a^2-1)^2}{4ab-1}$. Then $r = (-r)(-1) \equiv -r(4ab - 1) = -(4a^2 - 1)^2 \equiv -1 \pmod{4a}$ and $r = 4ac - 1$ for some positive integer c. From $a < b$ we obtain that

$$4ac - 1 = \frac{(4a^2 - 1)^2}{4ab - 1} < (4a^2 - 1)^2$$

and therefore $c < a$. By the construction, the number $4ac - 1$ is a divisor of $(4a^2 - 1)^2$, so (a, c) is a bad pair.

Property 2: If (a, b) is a bad pair then (b, a) is also bad.

Since $1 = 1^2 \equiv (4ab)^2 \pmod{4ab - 1}$, we have

$$(4b^2 - 1)^2 \equiv (4b^2 - (4ab)^2)^2 = 16b^4(4a^2 - 1)^2 \equiv 0 (\text{mod } 4ab - 1).$$

Hence the number $4ab - 1$ divides $(4b^2 - 1)^2$ as well.

Now suppose that there exists at least one bad pair. Take a bad pair (a, b) such that $2a + b$ attains its smallest possible value. If $a < b$ then Property 1 provides a pair (a, c) with $c < b$ and thus $2a + c < 2a + b$. Otherwise, if $b < a$, Property 2 yields that pair (b, a) is also bad while $2b + a < 2a + b$. Both cases contradict the assumption that $2a + b$ is minimal.

6. Answer: $3n$. It is easy to find such planes, e.g. $x = i, y = i, z = i (i = 1, 2, \ldots, n)$ or planes $x+y+z = k$ for $k = 1, 2, \ldots, 3n$. We show that $3n$ is the smallest possible number.

   Lemma 1. Consider a nonzero polynomial $P(x_1, \ldots, x_k)$ in k variables. Suppose that P vanishes at all points $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k \in \{0, 1, \ldots, n\}$ and $x_1 + \cdots + x_k > 0$, while $P(0, 0, \ldots, 0) \neq 0$. Then $\deg p \geq kn$.

   Proof of Lemma 1. We use induction on k. The base case $k = 0$ is clear since $P \neq 0$. Denote for clarity $y = x_k$.

   Let $R(x_1, \ldots, x_{k-1}, y)$ be the residue of P modulo $Q(y) = y(y-1)\cdots(y-n)$. Polynomial $Q(y)$ vanishes at each $y = 0, 1, \ldots, n$, hence $P(x_1, \ldots, x_{k-1}, y) = R(x_1, \ldots, x_{k-1}, y)$ for all $x_1, \ldots, x_{k-1}, y \in \{0, 1, \ldots, n\}$.

   Therefore R also satisfies the condition of the Lemma; moreover, $\deg_y R \leq n$. Clearly $\deg R \leq \deg P$, so it suffices to prove that $\deg R \geq nk$.

   Now expand polynomial R in the powers of y:

$$R(x_1, \ldots, x_{k-1}, y) = R_n(x_1, \ldots, x_{k-1})y^n + R_{n-1}(x_1, \ldots, x_{k-1})y^{n-1} + \cdots + R_0(x_1, \ldots, x_{k-1}).$$

   We show that polynomial $R_n(x_1, \ldots, x_{k-1})$ satisfies the condition of the induction hypothesis.

   Consider the polynomial $T(y) = R(0, \ldots, 0, y)$ of degree $\leq n$. This polynomial has n roots $y = 1, \ldots, n$; on the other hand, $T(y)$ is not identically 0 since $T(0) \neq 0$. Hence, $\deg T = n$ and its leading coefficient is $R_n(0, 0, \ldots, 0) \neq 0$. In particular in the case $k = 1$ we obtain that coefficient R_n is nonzero.

   Similarly, take any numbers $a_1, \ldots, a_{k-1} \in \{0, 1, \ldots, n\}$ with $a_1 + \cdots + a_{k-1} > 0$. Substituting $x_i = a_i$ into $R(x_1, \ldots, x_{k-1}, y)$, we get a polynomial in y which vanishes at all points $y = 0, \ldots, n$ and has degree $\leq n$. Therefore, this polynomial is null, hence $R_i(a_1, \ldots, a_{k-1}) = 0$ for all $i = 0, 1, \ldots, n$. In particular, $R_n(a_1, \ldots, a_{k-1}) = 0$.

   Thus the polynomial $R_n(a_1, \ldots, a_{k-1})$ satisfies the condition of the induction hypothesis. So we have $\deg R_n \geq (k-1)n$ and $\deg P \geq \deg R \geq \deg R_n + n \geq kn$, proving the Lemma.

   Now we can finish the solution. Suppose there are N planes covering all the points of S but not containing the origin. Let their equations be $a_i x + b_i y + c_i z + d_i = 0$. Consider the polynomial

$$P(x, y, z) = \prod_{i=i}^{N}(a_i x + b_i y + c_i z + d_i).$$

   It has total degree N. This polynomial has the property that $P(x_0, y_0, z_0) = 0$ for any $(x_0, y_0, z_0) \in S$ while $P(0, 0, 0) \neq 0$. Hence by Lemma 1 we have $N = \deg P \geq 3n$, as desired.