Solutions of the problems of IMO 2005 (MEXICO), by Giedrius Alkauskas.

Problem 1. First, draw a sketch. Let A be the top vertex of the equilateral triangle, and counter clock-wise mark B and C. Let the lenghts of B_2A, AC_1, C_2B, BA_1, A_2C and CB_1 be correspondingly a_1, a_2, b_1, b_2, c_1 and c_2. Let $|AC| = X$ and the sides of hexagon all have lengths l.
From the cosine theorem $a_1^2 + a_2^2 - a_1a_2 = l$, the same with $b's$ and $c's$, and, obviously, $a_1 + c_2 = c_1 + b_2 = b_1 + a_2 = X - l$. Suppose from the symmetry $a_1 \geq a_2$. Then $a_1 \geq l$, $a_2 \leq l$. If $b_2 > b_1$, then also $b_2 > l$, $b_1 < l$, and whence $c_1+b_2 = a_1+c_2 = b_1+a_2 < l+l = 2l$. In this case $c_1 < l$ and $c_2 \leq l$, which cannot occur.
Therefore, $b_1 \geq b_2$ and similarly $c_1 \geq c_2$. Changing the notation of a's, b's and c's, we can achieve $a_1 \geq b_1 \geq c_1$. Hence, we have ordered all values:

$$a_1 \geq b_1 \geq c_1 \geq b_2 \geq a_2 \geq c_2.$$

Now, $2l = 2(a_1^2 + a_2^2 - a_1a_2) = a_1^2 + a_2^2 + (a_1 - a_2)^2 \geq c_1^2 + c_2^2 + (c_1 - c_2)^2 = 2l$. In fact

$$X - l \geq X - l + (a_2 - b_2) \Rightarrow a_1 + c_2 \geq c_1 + b_2 + a_2 - b_2 \Rightarrow a_1 - a_2 \geq c_1 - c_2 \geq 0.$$

Hence, we should have equalities $a_1 = b_1 = c_1$ and $a_2 = b_2 = c_2$.
Whence trivially the triangle $A_1B_1C_1$ is equilateral also, and the required three lines are its bisectors. Proved.

Problem 2. Let the sequence a_1, a_2,... satisfy the given condition. Suppose, a distinct integer T appears at least twice. Then taking n sufficiently large, the set a_1, a_2,..., a_n will have two terms equal to T, and they will give two equal residues modulo n- a contradiction. Hence it is enough to show that each integer in fact appears.
Take first N terms of the sequence: a_1, a_2,...,a_N. Let M and m be the largest and correspondingly the smallest among these numbers. Then $M - m < N$, since otherwise, if $M - m = L \geq N$, then set a_1, a_2,...,a_L will have two terms with the same residue modulo L (these are $a_i = m$ and $a_j = M$). Hence a_1, a_2,...,a_N is some reordering of N consecutive integers.
Now let $A \in \mathbb{Z}$. Choose any $B < A$ and $C > A$, appearing in the given sequence (since it contains infinitely many positive and negative terms). Let N be such that both B and C appear in a_1, a_2,...,a_N. Then all numbers between the greatest and the smallest of these are contained in this set; in particular, all integers between B and C, and therefore A also. Proved.

Problem 3. ("Ugly" solution). Obviously, this inequality is equivalent to

$$\frac{1}{x^5 + y^2 + z^2} + \frac{1}{x^2 + y^5 + z^2} + \frac{1}{x^2 + y^2 + z^5} \leq \frac{3}{x^2 + y^2 + z^2}.$$

Naturally, it is enough to prove this for $xyz = 1$; in fact, changing $x = x't$, $y = y't$, $z = z't$ with $x'y'z' = 1$, $t \geq 1$, we need to prove

$$\frac{1}{t^3x^5 + y^2 + z^2} + \frac{1}{x^2 + t^3y^5 + z^2} + \frac{1}{x^2 + y^2 + t^3z^5} \leq \frac{3}{x^2 + y^2 + z^2},$$

and this suffice to prove for $t = 1$.
Let $x \leq y \leq z = w$, w be fixed. Let $w \geq 1.2$. In this case we will prove this inequality by "rude" method. Consider two cases.
(i) $y \leq 1$. Then it is easy to see that: $1-x^3 \leq 1-w^{-3}$, $1-y^3 \leq 1-w^{-3/2}$, $\frac{y^2+z^2}{x^2} \geq 1+w^3$, $\frac{z^2+x^2}{y^2} \geq w^2 + w^{-2}$, $\frac{x^2+y^2}{z^2} \leq w^{-2} + w^{-4}$.
Therefore,

$$\frac{x^2(1-x^3)}{x^5 + y^2 + z^2} \leq \frac{1 - w^{-3}}{w^{-3} + w^3 + 1},$$

$$\frac{y^2(1-y^3)}{y^5 + z^2 + x^2} \leq \frac{1 - w^{-3/2}}{w^{-3/2} + w^2 + w^{-2}},$$

$$\frac{z^2(z^3-1)}{z^5 + x^2 + y^2} \geq \frac{w^3 - 1}{w^3 + w^{-2} + w^{-4}}.$$

Therefore, it is enough to check that

$$\frac{1 - w^{-3}}{w^{-3} + w^3 + 1} + \frac{1 - w^{-3/2}}{w^{-3/2} + w^2 + w^{-2}} < \frac{w^3 - 1}{w^3 + w^{-2} + w^{-4}}$$

for these values of w. This inequality can be rewritten as

$$\frac{w^3}{w^6 + w^3 + 1} + \frac{w^3}{w^5 + w^{7/2} + w^{3/2} + w + 1 + w^{-1/2}} < \frac{w^7}{w^7 + w^2 + 1}.$$

Since by arithmetic - geometric mean inequality $w^{7/2}+w^{3/2}+w+1+w^{-1/2} \geq 5w^{1.1} > 5w$, the second summand is $< \frac{w^3}{w^5+5w} = \frac{w^2}{w^4+5}$. This function acquires maximum at $w = \sqrt[4]{5}$, which is equal to $\sqrt{5}/10$. The first summand acquires maximum at $w = 1$, which is $1/3$. In total, left hand side is $\leq \sqrt{5}/10+1/3 < 0.56$. The right hand side is increasing function for $w \geq 1$. Its value at $w = 1.2$ is > 0.59. Hence, provided that $w \geq 1.2$, the inequality holds.
(ii) $y \geq 1$. In the same fashion, $\frac{x^2+y^2}{z^2} \leq 1 + w^{-6}$, $\frac{y^2+z^2}{x^2} \geq w^2 + w^4$, $1 - x^3 \leq 1 - w^{-6}$.
Therefore

$$\frac{x^2(1-x^3)}{x^5 + y^2 + z^2} \leq \frac{1 - w^{-6}}{w^{-6} + w^2 + w^4},$$

$$\frac{y^2(y^3-1)}{y^5 + z^2 + x^2} \geq 0,$$

$$\frac{z^2(z^3-1)}{z^5 + x^2 + y^2} \geq \frac{w^3 - 1}{w^3 + 1 + w^{-6}}.$$

Hence, it is enough to check that

$$\frac{w^3 - 1}{w^3 + 1 + w^{-6}} \geq \frac{1 - w^{-6}}{w^{-6} + w^2 + w^4}.$$

This is a consequence of $w^3 + w^{-6} \geq 2$, for $w \geq 1.2$ and $w^4 + w^2 \geq w^3 + 1$ for $w \geq 1.2$. Therefore, we may assume that the maximal of x, y, z is less than or equal to 1.2; that is $0.69 \leq x, y, z \leq 1.2$.

The given inequality is equivalent to

$$\sum_{v=x,y,z} \frac{1}{A + v^5 - v^2} \leq \frac{3}{A},$$

where $A = x^2 + y^2 + z^2$. This can be written as

$$\sum_{\lambda=\log x, \log y, \log z} \frac{1}{A + e^{5\lambda} - e^{2\lambda}} \leq \frac{3}{A}.$$

Define a function $F(\lambda) = (A + e^{5\lambda} - e^{2\lambda})^{-1}$. Note that $\log x + \log y + \log z = 0$. If the function $F(\lambda)$ is cap-concave in the interval $(\log x, \log z)$, then this inequality follows directly from Jensen's inequality:

$$\frac{1}{3}\Big(F(\lambda_1) + F(\lambda_2) + F(\lambda_3)\Big) \leq F\Big(\frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)\Big)$$

(minding that $F(0) = 1/A$).
Hence, in order to finish, we will check that this is the case. The inequality $F''(\lambda) < 0$ is equivalent to

$$A(25e^{5\lambda} - 4e^{2\lambda}) + 11e^{7\lambda} > 25e^{10\lambda} + 4e^{4\lambda}.$$

In our case, $\lambda \leq \log z$. We have $A = x^2 + y^2 + z^2 \geq z^2 + \frac{2}{z}$. Since the function $e^{2s} + 2e^{-s}$ is increasing for $s \geq 1$, we have

$$A \geq e^{2\lambda} + 2e^{-\lambda}.$$

Therefore, it is enough to prove that $(e^{2\lambda} + 2e^{-\lambda})(25e^{5\lambda} - 4e^{2\lambda}) + 11e^{7\lambda} > 25e^{10\lambda} + 4e^{4\lambda}$, which is equivalent to

$$42e^{3\lambda} + 36e^{6\lambda} < 25e^{9\lambda} + 8.$$

If we denote $e^{3\lambda}$ by W, then it is equivalent to

$$25W^3 - 36W^2 - 42W + 8 < 0.$$

We can check easily that one root of this cubic polynomial is negative, the other is between 0 and 0.2, another is > 2. Since our range is $0.69^3 > 0.2$ and $1.2^3 < 2$, this inequality

holds, and WE ARE DONE!
Since function $F(\lambda)$ is strictly cap-concave in our interval, the equality applying Jensen's inequality is achieved only if $x = y = z = 1$.

Problem 4. Trivially, $a_2 = 48$ is divisible by 2 and 3. Let therefore $p > 3$ be a prime number. Then

$$6a_{p-2} = 3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} - 6 \equiv 3 + 2 + 1 - 6 \equiv 0 \pmod{p}.$$

And so the only such number is 1. Proved.

Problem 5. First, draw a sketch. Let A be the top left vertex, and mark clockwise B, C and D. (In our sketch Q and R belong to intervals DP and CP).
We will show that the centers of these circumcircles all lie on a certain fixed line. Hence the point P', symmetric to P with respect to this line, will be the needed point (it is easy to verify that P does not belong to this line).
From sine theorem for $\triangle FQD$ and $\triangle BQE$ we have:

$$\frac{FQ}{\sin(\angle ADB)} = \frac{FD}{\sin(\angle FQD)} = \frac{BE}{\sin(\angle BQE)} = \frac{QE}{\sin(\angle DBC)}.$$

Therefore, $FQ : QE$ is fixed ratio. Similarly, $FR : RE$ is constant, and whence $FQ : QR : RE$ is fixed ratio, not depending on the choice of F and E.
Let O be the center of the circumcircle of $\triangle PQR$. Then, since $\angle QPR$ is fixed, the $\angle QOR$ is also fixed, and the perpendicular $OS \perp QR$ is of length proportional to the length of QR. Since $QS = SR$, we finally obtain that triangles $\triangle FOE$ are all similar.
We will finish employing complex numbers. Embed our triangle into complex plane. Let $D = z_1$, $B = z_2$, $\frac{A-D}{|A-D|} = \xi_1$, $\frac{C-B}{|C-B|} = \xi_2$. Then $|\xi_1| = |\xi_2| = 1$. The point $F = z_1 + \alpha\xi_1$, $E = z_2 + \alpha\xi_2$ with $\alpha \in \mathbb{R}$. As we have proved, the expression $\frac{O-F}{E-F}$ is a constant complex number ω. Therefore:

$$O = z_1 + \alpha\xi_1 + \omega(z_2 + \alpha\xi_2 - z_1 - \alpha\xi_1) = (z_1 + \omega z_2 - \omega z_1) + \alpha(\xi_1 + \omega\xi_2 - \omega\xi_1).$$

Both brackets are certain fixed complex numbers u and v. Whence $O = u + \alpha v$, $\alpha \in \mathbb{R}$, and this is a line in the complex plane.
The multiplier at α cannot be 0, since then O itself is fixed, and we get a parallelogram. Proved.

Problem 6. Let N be the number of students. Suppose, no one solved five problems. For each student, count all pairs of problems he solved, and then add. On one hand, everyone solved $\leq \frac{4 \cdot 3}{2} = 6$ pairs, hence totally $\leq 6N$ pairs. On the other, each pair was solved by $> \frac{2}{5}N$ students, hence totally $> \frac{6 \cdot 5}{2}\frac{2}{5}N = 6N$ - a contradiction.
Suppose, exactly one student solved 5 problems
(...Zinau, kaip sprest, tik tingiu surasinet...)