=== CIRCLE ===

Equations of the Common Tangents to Two Circles

By:

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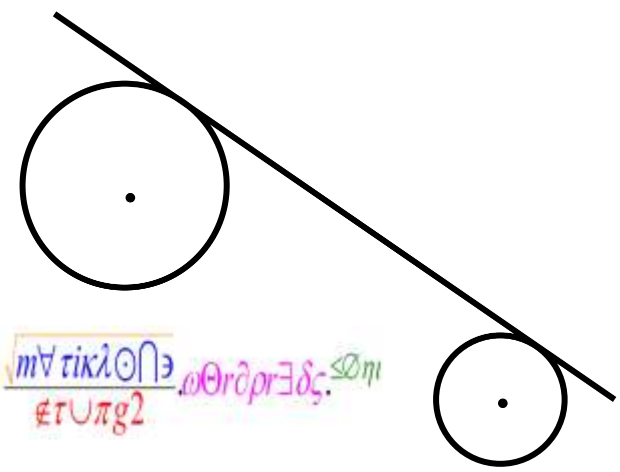

MatikZone's Series

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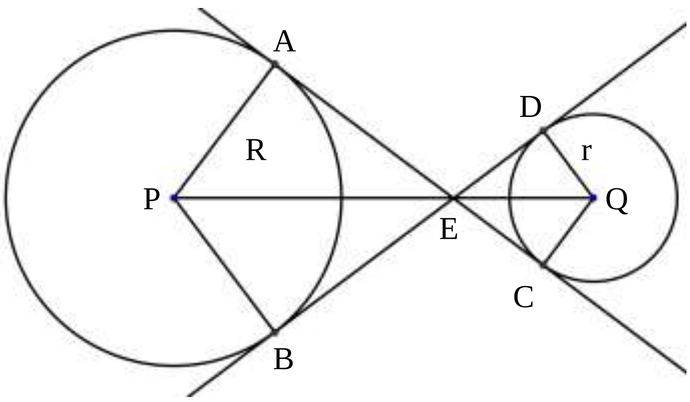
Equations of the Common Tangents to Two Circles

Let:
$P = P(x_P, y_P)$ **= Center of first circle or** L_1
$Q = Q(x_Q, y_Q)$ **= Center of second circle or** L_2
R = **Radius of first circle or** L_1
r = **radius of second circle or** L_2
E = **intersection poin to interior common tangents**
S = **intersection poin to exterior common tangents**

INTERIOR COMMON TANGENTS

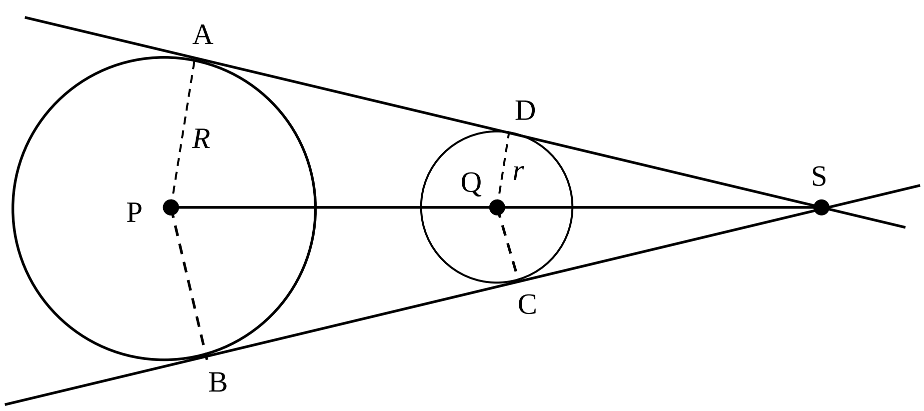


$\Delta PBE \sim \Delta QDE$, because $\angle PBE = \angle QDE = 90^0$ and $\angle PEB = \angle QED$ then $\angle BPE = \angle DQE$,

we have $\dfrac{PE}{QE} = \dfrac{PB}{QD} = \dfrac{R}{r}$ or $PE : QE = R : r$. (E dividing PQ with ratio $PE : QE = R : r$)

The coordinate E is $E(x_E,\ y_E) = E\left(\dfrac{Rx_Q + rx_P}{R+r},\ \dfrac{Ry_Q + ry_P}{R+r}\right)$

EXTERIOR COMMON TANGENTS



$\Delta PBS \sim \Delta QCS$, because $\angle PBS = \angle QCS = 90^0$ and $\angle PSB = \angle QSC$ then $\angle BPS = \angle CQS$, we have $\frac{PQ+QS}{QS} = \frac{PB}{QC} = \frac{R}{r} \Rightarrow \frac{PQ}{QS} + 1 = \frac{R}{r} \quad \Rightarrow \frac{PQ}{QS} = \frac{R-r}{r};\ R > r$

(Q dividing PS with ratio $PQ : QS = (R-r) : r;\ R > r$), then

$$Q(x_Q,\ y_Q) = Q\left(\frac{(R-r)x_S + rx_P}{(R-r)+r},\ \frac{(R-r)y_S + ry_P}{(R-r)+r}\right)$$

$$\Rightarrow x_Q = \frac{(R-r)x_S + rx_P}{(R-r)+r} \Rightarrow \qquad Rx_Q = (R-r)x_S + rx_P$$

$$\Rightarrow (R-r)x_S = Rx_Q - rx_P$$

$$\Rightarrow \qquad x_S = \frac{Rx_Q - rx_P}{(R-r)}$$

and

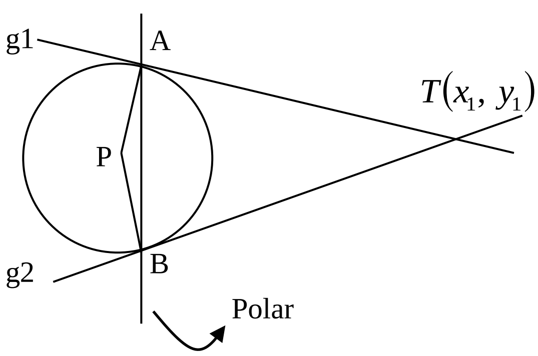
$$\Rightarrow y_Q = \frac{(R-r)y_S + ry_P}{(R-r)+r} \Rightarrow \qquad Ry_Q = (R-r)y_S + ry_P$$

$$\Rightarrow (R-r)y_S = Ry_Q - ry_P$$

$$\Rightarrow \qquad y_S = \frac{Ry_Q - ry_P}{(R-r)}$$

So, the coordinate S is $S(x_S,\ y_S) = S\left(\frac{Rx_Q - rx_P}{R-r},\ \frac{Ry_Q - ry_P}{R-r}\right)$

To determine the equations of the common tangents, follow this instructions:



1. Determine the polar equation
2. Substitute to the circle equation to find point A and point B coordinate.
3. Use the equation tangent pass trough a point on the circle. The equation is same as the polar equation.

Circle: $(x - x_P)^2 + (y - y_P)^2 = R^2$,

polar equation: $(x_1 - x_P)(x - x_P) + (y_1 - y_P)(y - y_P) = R^2$

Circle: $x^2 + y^2 + Ax + By + C = 0$,

polar equation: $x_1 x + y_1 y + \frac{A}{2}(x_1 + x) + \frac{B}{2}(y_1 + y) + C = 0$

EXTERIOR COMMON TANGENTS, if $R = r$.

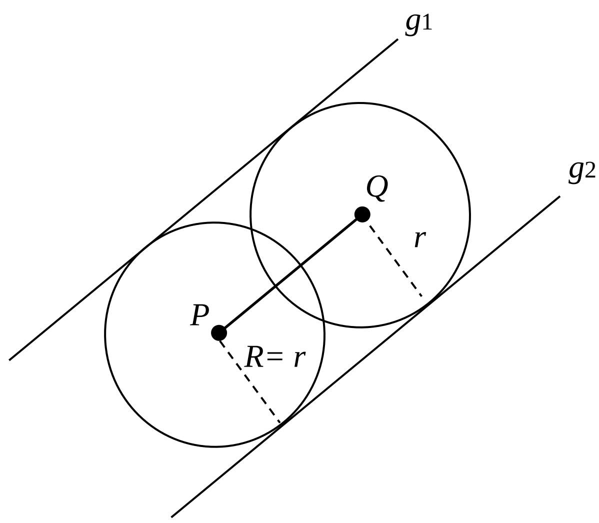
If $R = r$ we have $m_g = m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$

The equations of tangent is:

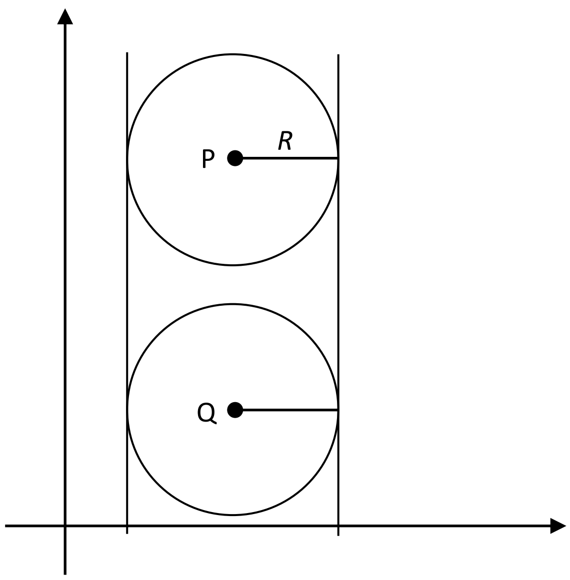
$$y - y_P = m_g(x - x_P) \pm R\sqrt{1 + m_g^2}$$

or

$$y - y_Q = m_g(x - x_Q) \pm r\sqrt{1 + m_g^2}$$



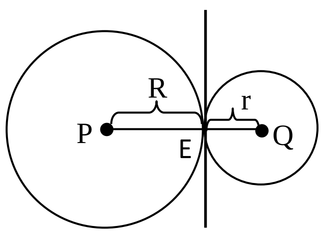
EXTERIOR COMMON TANGENTS, if $R = r$ and $x_P = x_Q$.



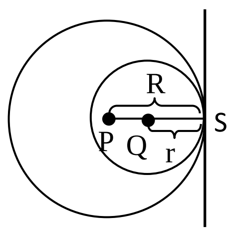
The equations tangent line is:

$x = x_P + R$ and $x = x_P - R$

TOUCHING CIRCLES



2 Circles touch externally



2 Circles touch internally

$E\left(\frac{Rx_Q + rx_P}{R+r}, \frac{Ry_Q + ry_P}{R+r}\right)$ and $S\left(\frac{Rx_Q - rx_P}{R-r}, \frac{Ry_Q - ry_P}{R-r}\right)$ is the point of contact, so the equation of the common tangents is:

2 Circles touch externally:
$(x_E - x_P)\,(x - x_P) + (y_E - y_P)\,(y - y_P) = R^2$ or
$(x_E - x_Q)(x - x_Q) + (y_E - y_Q)(y - y_Q) = r^2$

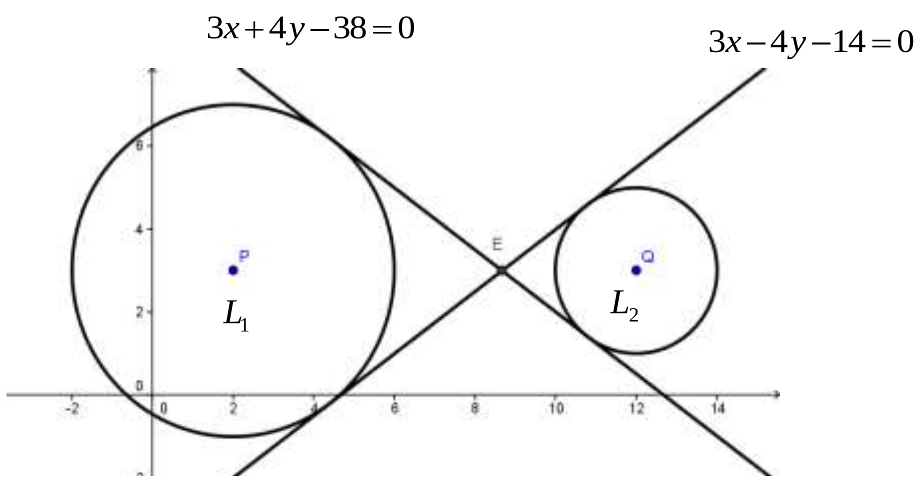
2 Circles touch internally
$(x_S - x_P)\,(x - x_P) + (y_S - y_P)\,(y - y_P) = R^2$ or
$(x_S - x_Q)(x - x_Q) + (y_S - y_Q)(y - y_Q) = r^2$

EXAMPLE:

Problem 1:

Find the equations of the interior common tangents to $L_1 \equiv (x-2)^2 + (y-3)^2 = 16$ and $L_2 \equiv (x-12)^2 + (y-3)^2 = 4$.

Solusion:



$L_1 \equiv (x-2)^2 + (y-3)^2 = 16$ with center P(2, 3) and radius $R = 4$
$L_2 \equiv (x-12)^2 + (y-3)^2 = 4$ with center Q(12, 3) and radius $r = 2$

The relationship of 2 circles

$$\left.\begin{array}{l} PQ = \sqrt{(12-2)^2 + (3-3)^2} = \sqrt{100} = 10 \\ R + r = 4 + 2 = 6 \\ R - r = 4 - 2 = 2 \end{array}\right\} R + r < PQ \text{ and } R - r < PQ$$

The relationship is non intersecpting externally, there are 2 interior common tangents.

$$E\left(\frac{4.12 + 2.2}{4+2},\ \frac{4.3 + 2.3}{4+2}\right) = E\left(\frac{52}{6},\ \frac{18}{6}\right) = E\left(\frac{26}{3},\ 3\right)$$

Method 1: using L_1.

The equations of tangent to cirle 1 with gradient *m* is:

$$y-y_P=m(x-x_P)\pm R\sqrt{1+m^2}\Rightarrow y-3=m(x-2)\pm 4\sqrt{1+m^2}$$

The tangent pass trough $E\left(\frac{26}{3},\ 3\right)$, so

$$y-3=m(x-2)\pm 4\sqrt{1+m^2}\Rightarrow \qquad 3-3=m\left(\frac{26}{3}-2\right)\pm 4\sqrt{1+m^2}$$

$$\Rightarrow \qquad 0=\frac{20}{3}m\pm 4\sqrt{1+m^2}$$

$$\Rightarrow \pm 4\sqrt{1+m^2}=\frac{20}{3}m$$

$$\Rightarrow 16+16m^2=\frac{400m^2}{9}$$

$$\Rightarrow 144+144m^2=400m^2$$

$$\Rightarrow 256m^2=144$$

$$\Rightarrow 16m^2=9$$

$$\Rightarrow m^2=\frac{9}{16}$$

$$\Rightarrow m=\pm\frac{3}{4}$$

for $m=\frac{3}{4}\Rightarrow \qquad y-3=\frac{3}{4}\left(x-\frac{26}{3}\right)$

$\Rightarrow \qquad y-3=\frac{3}{4}x-\frac{26}{4}$

$\Rightarrow 3x-4y-14=0$

for $m=-\frac{3}{4}\Rightarrow \qquad y-3=-\frac{3}{4}\left(x-\frac{26}{3}\right)$

$\Rightarrow \qquad y-3=-\frac{3}{4}x+\frac{26}{4}$

$\Rightarrow 3x+4y-38=0$

The equations of the interior common tangents are:

- $g_1\equiv 3x-4y-14=0$
- $g_2\equiv 3x+4y-38=0$

Method 2: using L_2.

The equations of tangent to cirle 2 with gradient *m* is:

$$y-y_Q=m(x-x_Q)\pm r\sqrt{1+m^2}\Rightarrow y-3=m(x-12)\pm 2\sqrt{1+m^2}$$

The tangent pass trough $E\left(\frac{26}{3},\ 3\right)$

$$y-3=m(x-12)\pm 2\sqrt{1+m^2} \Rightarrow \qquad 3-3=m\left(\frac{26}{3}-12\right)\pm 2\sqrt{1+m^2}$$

$$\Rightarrow \qquad 0=-\frac{10}{3}m\pm 2\sqrt{1+m^2}$$

$$\Rightarrow \quad \pm 2\sqrt{1+m^2}=-\frac{10}{3}m$$

$$\Rightarrow \quad 4+4m^2=\frac{100m^2}{9}$$

$$\Rightarrow \quad 36+36m^2=100m^2$$

$$\Rightarrow \quad 64m^2=36$$

$$\Rightarrow \quad 16m^2=9$$

$$\Rightarrow \quad m^2=\frac{9}{16}$$

$$\Rightarrow \quad m=\pm\frac{3}{4}$$

for $m=\frac{3}{4} \Rightarrow \qquad y-3=\frac{3}{4}\left(x-\frac{26}{3}\right)$

$$\Rightarrow \qquad y-3=\frac{3}{4}x-\frac{26}{4}$$

$$\Rightarrow 3x-4y-14=0$$

for $m=-\frac{3}{4} \Rightarrow \qquad y-3=-\frac{3}{4}\left(x-\frac{26}{3}\right)$

$$\Rightarrow \qquad y-3=-\frac{3}{4}x+\frac{26}{4}$$

$$\Rightarrow 3x+4y-38=0$$

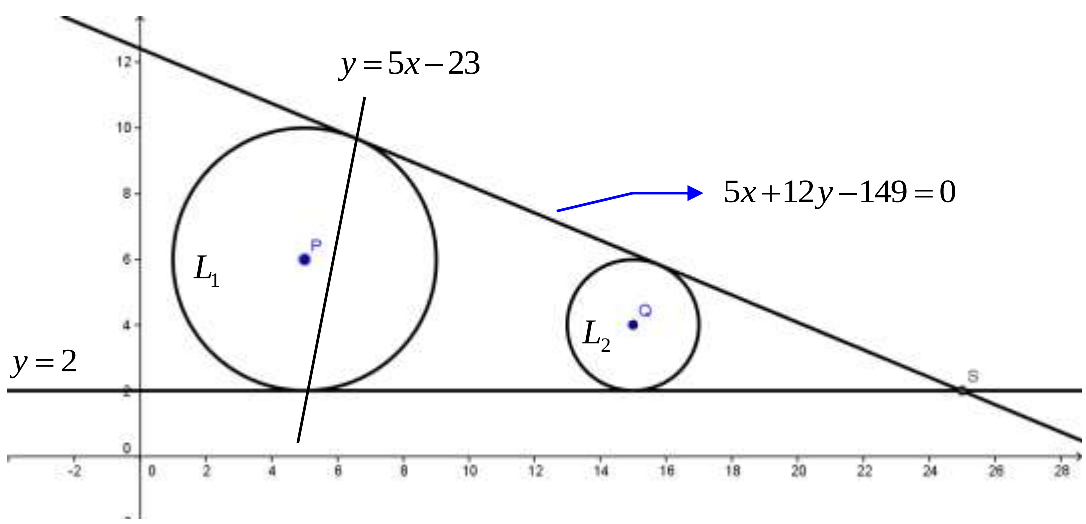
The equations of the interior common tangents are:

- $g_1 \equiv 3x-4y-14=0$
- $g_2 \equiv 3x+4y-38=0$

Problem 2:

Find the equations of the exterior common tangents to $L_1 \equiv (x-5)^2+(y-6)^2=16$ and $L_2 \equiv (x-15)^2+(y-4)^2=4$.

Solution:



$L_1 \equiv (x-5)^2 + (y-6)^2 = 16$, the center P(5, 6) and radius $R = 4$
$L_2 \equiv (x-15)^2 + (y-4)^2 = 4$, the center Q(15, 4) and radius $r = 2$

The relationship of 2 circles

$$\left.\begin{array}{l} PQ = \sqrt{(15-5)^2 + (4-6)^2} = \sqrt{100+4} = \sqrt{104} \\ R + r = 4 + 2 = 6 \\ R - r = 4 - 2 = 2 \end{array}\right\} R + r < PQ \text{ and } R - r < PQ$$

The relationship is non intersecpting externally, there are 2 exterior common tangents

$$S\left(\frac{4.15-2.5}{4-2},\ \frac{4.4-2.6}{4-2}\right) = S\left(\frac{50}{2},\ \frac{4}{2}\right) = S(25,\ 2)$$

Method 1: using Polar and L1

The polar from S(25, 2) on circle 1 (L1) is:

$$(x_E - x_P)\,(x - x_P) + (y_E - y_P)\,(y - y_P) = R^2$$

$$\begin{aligned} &\Rightarrow (25-5)(x-5) + (2-6)(y-6) = 16 \\ &\Rightarrow 20x - 100 - 4y + 24 - 16 = 0 \\ &\Rightarrow 20x - 4y - 92 = 0 \\ &\Rightarrow y = 5x - 23 \end{aligned}$$

Subtitute to L1

$$\begin{aligned} (x-5)^2 + (y-6)^2 = 16 &\Rightarrow (x-5)^2 + (5x-29)^2 = 16 \\ &\Rightarrow x^2 - 10x + 25 + 25x^2 - 290x + 841 - 16 = 0 \\ &\Rightarrow 26x^2 - 300x + 850 = 0 \\ &\Rightarrow 13x^2 - 150x + 425 = 0 \\ &\Rightarrow (13x-85)(x-5) = 0 \\ &\Rightarrow x = \frac{85}{13} \text{ or } x = 5 \end{aligned}$$

Subtitute x to polar equation.

$$* x = \frac{85}{13} \Rightarrow y = 5\cdot\frac{85}{13} - 23 = \frac{425}{13} - \frac{299}{13} = \frac{126}{13} \qquad \Rightarrow T_1\left(\frac{85}{13},\ \frac{126}{13}\right)$$

$$* x = 5 \ \Rightarrow y = 5\cdot 5 - 23 = 25 - 23 = 2 \qquad \Rightarrow T_2(5,\ 2)$$

$T_1\left(\frac{85}{13},\ \frac{126}{13}\right)$ and $T_2(5,\ 2)$ is the contact point, so the equations of tangents are:

:

$$T_1\left(\frac{85}{13},\ \frac{126}{13}\right) \Rightarrow \left(\frac{85}{13}-5\right)(x-5)+\left(\frac{126}{13}-6\right)(y-6)=16$$

$$\Rightarrow \quad \frac{20}{13}(x-5)+\frac{48}{13}(y-6)-16=0$$

$$\Rightarrow \quad 20(x-5)+48(y-6)-208=0$$

$$\Rightarrow \quad 20x+-100+48y-288-208=0$$

$$\Rightarrow \quad 20x+48y-596=0$$

$$\Rightarrow \quad 5x+12y-149=0$$

$$T_1(5,\ 2) \Rightarrow (5-5)(x-5)+(2-6)(y-6)=16$$

$$\Rightarrow \quad 0(x-5)-4(y-6)-16=0$$

$$\Rightarrow \quad -4y+24-16=0$$

$$\Rightarrow \quad -4y=-8$$

$$\Rightarrow \quad y=2$$

Method 2: using gradient and L1.

The equations of tangent to cirle 1 (L1) with gradient *m* is:

$$y-y_P=m(x-x_P)\pm R\sqrt{1+m^2} \Rightarrow y-6=m(x-5)\pm 4\sqrt{1+m^2}$$

The tangent pass trough $S(25,\ 2)$

$$y-6=m(x-5)\pm 4\sqrt{1+m^2} \Rightarrow \quad 2-6=m(25-5)\pm 4\sqrt{1+m^2}$$

$$\Rightarrow \quad -4=20m\pm 4\sqrt{1+m^2}$$

$$\Rightarrow \quad -1=5m\pm\sqrt{1+m^2}$$

$$\Rightarrow \quad \pm\sqrt{1+m^2}=5m+1$$

$$\Rightarrow \quad 1+m^2=25m^2+10m+1$$

$$\Rightarrow \quad 24m^2+10m=0$$

$$\Rightarrow \quad m(24m+10)=0$$

$$\Rightarrow \quad m=0 \quad \text{or} \quad m=-\frac{10}{24}=-\frac{5}{12}$$

for $m=0 \Rightarrow \quad y-2=0(x-25)$

$$\Rightarrow \quad y-2=0$$

$$\Rightarrow \quad y=2$$

for $m=-\frac{5}{12} \Rightarrow \quad y-2=-\frac{5}{12}(x-25)$

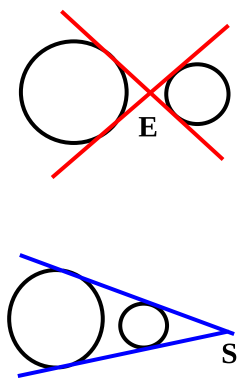
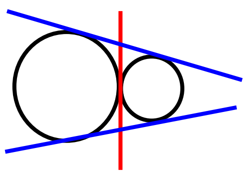
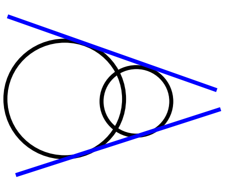
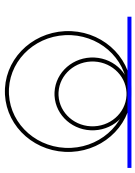
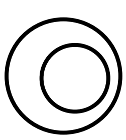
$$\Rightarrow \quad y-2=-\frac{5}{12}x+\frac{125}{12}$$

$$\Rightarrow \quad 12y-24=-5x+125$$

$$\Rightarrow \quad 5x+12y-149=0$$

The equations of the exterior common tangents are: $y=2$ and $5x+12y-149=0$

Common Tangents to Two Circles

2 Circles Relasionship		Number of common tangent		How to Determine the Equation	
		int	**ext**	**Interior common tangent**	**Exterior common tangent**
PQ > R + r	 Non intersecting externally	2	2	• Determine E • Determine the tangent pass trough point outside the circle. $E\left(\frac{Rx_Q+rx_P}{R+r}, \frac{Ry_Q+ry_P}{R+r}\right)$ note: L1: center P , radius R L2: center Q, radius r	• Determine S • Determine the tangent pass trough point outside the circle. $S\left(\frac{Rx_Q-rx_P}{R-r}, \frac{Ry_Q-ry_P}{R-r}\right)$ If R = r, use equation tangent with gradient m $m_g = m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$ If R = r and $x_P = x_Q$, the common tangent are $x = x_p \pm R$
PQ = R + r	 Touching Externally	1	2	• Determine E • Determine the tangent pass trough point on the circle **or** $L_1 - L_2 = 0$	-- Same as before --
|R – r| < PQ < R + r	 Intersecting Circle	0	2	-	-- Same as before --
PQ = |R – r|	 Touching Internally	0	1	-	• Determine S • Determine the tangent pass trough point on the circle **or** $L_1 - L_2 = 0$
PQ < |R – r|	 Non intersecting internally	0	0	-	-